Grade 11/12 Math Circles
October 25, 2023
P-adic numbers, Part 1 - Solutions

Exercise Solutions

Exercise 1

Find a 10-adic number that equals $\frac{1}{3}$.

Exercise 1 Solution

This number multiplied by 3 should also be $\ldots 0000000001$, we can reconstruct as

$$\begin{array}{r} \ldots ??????????7 \\ \times\ \underline{\ldots 0000000003} \\ \ldots 0000000021 \end{array}$$

Then we pick next digit to get $\ldots 01$:

$$\begin{array}{r} \ldots ?????????67 \\ \times\ \underline{\ldots 0000000003} \\ \ldots 0000000021 \\ +\ \underline{\ldots 0000000180} \\ \ldots 0000000201 \end{array}$$

Continue in the same fashion to get $\ldots 6666667 = \frac{1}{3}$.

Exercise 2

Find a 10-adic number that equals -3.

Exercise 2 Solution

This number when added with 3 should result in $\ldots 0000000000$, so

$$\begin{array}{r} \ldots 9999999997 \\ + \underline{\ldots 0000000003} \\ \ldots 0000000000 \end{array}$$

Then this process should converge on the number $\ldots 9999999997 = -3$.

Exercise 3

How would you prove Theorem 1?

Exercise 3 Solution

A quick, yet informal proof can be done by noticing that the sum of complements in any p-adic system is equal to $p-1$, then the digit corresponding to this number is $p-1$, which for 10-adics is simply 9, finally if we just took complement and sum the two numbers x to $-x$, the result would be $\ldots 99999999$, then adding 1 would always result in 0, which is exactly what we wantfrom adding x to $-x$. The proof is similar for any p.

Exercise 4

Find a 3-adic expansion of -1.

Exercise 4 Solution

Similarly to Exercise 3, we can use an analog of Theorem 1 to get

$$\begin{array}{r} \ldots 2222222222 \\ + \underline{\ldots 0000000001} \\ \ldots 0000000000 \end{array}$$

Then this process should converge on the number $\ldots 2222222222 = -1$ in 3-adics.

Exercise 5

Find the first three digits of $\sqrt{7}$ in the 3-adic integers.

Exercise 5 Solution

First, we need to write down 7 in 3-adics:
$7 = \ldots 21 = 21_3$. Then we the number $\sqrt{7}$ from multiplication of a 3-adic number by itself, that should result in 21_3:

$$\begin{array}{r} \ldots ????????111 \\ \times \underline{\ldots ????????111} \\ \ldots 0000000111 \\ \ldots 0000001110 \\ + \underline{\ldots 0000011100} \\ \ldots 0000010021 \end{array}$$

So, $\sqrt{7} = \ldots 111_3$.

Exercise 6

Find the first few digits of $\sqrt{17}$ in the 2-adic integers.

Exercise 6 Solution

First, we need to write down 17 in 2-adics:
$17 = \ldots 10001 = 10001_2$. Then we the number $\sqrt{17}$ from multiplication of a 2-adic number by itself, that should result in 10001_2:

$$\begin{array}{r} \ldots ???????1011 \\ \times \; \underline{\ldots ???????1011} \\ \ldots 0000001011 \\ \ldots 0000010110 \\ \ldots 0000000000 \\ + \; \underline{\ldots 0001011000} \\ \ldots 0001110001 \end{array}$$

So, $\sqrt{17} = \ldots 1011_2$.

Exercise 7

Find p for which the equation $x^2 = -1$ has at least a single solution.
Hint:
Check $p = 2, 3$ first, then consider $\lim_{n\to\infty} 2^{5^n}$ in a 5-adic system.

Exercise 7 Solution

Cheking $p = 2$. Since $x^2 = -1$, we first write what -1 is on a 2-adic system:

$$\begin{array}{r} \ldots 1111111111 \\ + \; \underline{\ldots 0000000001} \\ \ldots 0000000000 \end{array}$$

Then this process should converge on the number $\ldots 1111111111 = -1$ in 2-adics.

Now we are looking for such a x, that $x^2 = \dots 1111111111$, which means

$$\begin{array}{r} \dots ??????????? \\ \times\ \underline{\dots ???????????} \\ \dots 1111111111 \end{array}$$

We only have two digits 0 and 1, the last digit can't be 0 otherwise the last digit of the product would have been 0, then it's 1:

$$\begin{array}{r} \dots ??????????1 \\ \times\ \underline{\dots ??????????1} \\ \dots 1111111111 \end{array}$$

Then the next digit can be either 0 or 1, again for same reasons, 0 is not a choice, however

$$\begin{array}{r} \dots ?????????11 \\ \times\ \underline{\dots ?????????11} \\ \dots ?????????01 \end{array}$$

Which means there is no such nu,ber x in 2-adics, such that $x^2 = -1$!
Cheking $p = 3$ is done in a similar way.
Cheking $p = 5$. Consider $\lim_{n\to\infty} 2^{5^n}$ in a 5-adic system. Remember the possible digits in this sytem are $0, 1, 2, 3, 4$.
The limit is a p-adic number, let's see how it looks like:

$$\begin{aligned} 2^1 &= 2_5, \\ 2^5 &= 32_5, \\ 2^{25} &= 33554432_5, \\ 2^{78125} &= \dots 413014\textcolor{red}{32431212}_5, \\ 2^{390625} &= \dots 010120\textcolor{red}{32431212}_5 \end{aligned}$$

if you found less digits, it is just fine, I used Wolfram to get these digits.
Let's call this limit $x = \lim_{n\to\infty} 2^{5^n} = \dots 32431212_5$

Since the limit is converging on a single number x, there is no difference in 5-adic number between 2^{5^n} and $2^{5^{n+1}}$ as $n \to \infty$.
Therefore $x^5 = x$.
Now let's square x, like we did before:

$$\begin{array}{r} \ldots 32431212 \\ \times \; \underline{\ldots 32431212} \\ \ldots 44444444 \end{array}$$

When we add 1 to the number $\ldots 44444444_5$ we get 0, so $\ldots 44444444_5 = -1_5$. This means that $x^2 = \ldots 44444444_5 = -1$!
This can also be seen from factorization of $x^5 - x = 0$:

$$x^5 - x = (x-1)x(x+1)(x^2+1) = 0$$

Problem Set Solutions

1. What's $\ldots 13131313_5 = ?$

Solution: Recall $\ldots 444444444 = -1_5$, let $x = \ldots 13131313_5$, then

$$100x = \ldots 13131300,$$

$$100x + 13_5 = \ldots 13131313 = x,$$

so $100x + 13 = x$, then $x = \frac{13}{1-100} = -\frac{13_5}{99_5} = -\frac{8}{24} = -\frac{1}{3}$ in base 10.

2. Find the numbers $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$ as real numbers and as 10-adic numbers. What do you notice?

Solution: In real numbers we can compute

$$\begin{aligned}1/7 &= 0.142857142857\ldots,\\ 2/7 &= 0.285714285714\ldots,\\ 3/7 &= 0.428571428571\ldots,\\ 4/7 &= 0.571428571429\ldots,\\ 5/7 &= 0.714285714286\ldots,\\ 6/7 &= 0.857142857143\ldots,\end{aligned}$$

in 10-adic numbers we have

$$\begin{aligned}1/7 &= \ldots 857142857143,\\ 2/7 &= \ldots 714285714286,\\ 3/7 &= \ldots 571428571429,\\ 4/7 &= \ldots 429571428572,\\ 5/7 &= \ldots 286714285715,\\ 6/7 &= \ldots 143857142858\end{aligned}$$

The 10-adic fractions are shifted and the have a +1 in the last digits.

3. Using the following theorem: *A p-adic number has an eventually periodic p-adic expansion if and only if it is rational, i.e. can be written as a fraction.* Determine the periodic 5-adic expansion of $\frac{4}{3}$.

Solution: We start by noticing that 4 is just 4_5 in 5-adic system and $\frac{1}{3} = \ldots 42424243_5$ by Theorem 1.

Then $\frac{4}{3} = 4_5 \times \ldots 42424243_5$, which is

$$\begin{array}{r} \ldots 424242423 \\ \times \underline{\ldots 000000004} \\ \ldots 131313132 \end{array}$$

So you can say the periodic 5-adic expansion of $\frac{4}{3}$ is periodic with (13) in period.

4. Show that a 2-adic integer that is a unit has a square root if and only if its last 3 digits are $\ldots 001$.

Solution: Proving the statement the last 3 digits of a 2-adic number are $\ldots 001$, then it can be a square root of unity, since

$$\begin{array}{r} \ldots ???????001 \\ \times \underline{\ldots ???????001} \\ \ldots ???????001 \end{array}$$

and all other digits would not result in $\ldots 001$.
The other way around if a 2-adic integer is a unit, then the last 3 digits of its square root are $\ldots 001$: